Int. J. Heat Mass Transfer. Vol. 26, No. 1, pp. 121–132, 1983
Printed in Great Britain

0017-9310/83/010121-12 $03.00/0

APPLICATION OF FOURIER TRANSFORMS TO PERIODIC HEAT FLOW INTO THE GROUND UNDER A BUILDING

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(*Received* 8 *February* 1982 *and in final form* 11 *June* 1982)

Abstract—Fourier transforms are used to obtain expressions for the heat flux from a surface region at a given temperature into a 2 or 3 dim. semi-infinite solid, with all quantities assumed to be periodic in time. These expressions are evaluated explicitly in the 2-dim. case and are used to obtain approximate results in 3-dim. valid for a wide range of frequencies. An explicit exact expression for the 3-dim. steady-state heat flux from a rectangular surface region is also obtained.

NOMENCLATURE

A,	area [m^2];
$B(z)$,	function defined in Section 3.1;
B,	constant defined in Section 1;
D,	distance [m];
$F_{x,y}$, F_x,	Fourier transform operators;
$G(\alpha,\beta)$,	function defined in Section 5;
$I(x)$,	function defined in Section 3.1;
$J_0(r\rho)$,	Bessel function of order 0;
$K_\nu(z)$,	modified Bessel function of order ν;
$Ki_r(x)$,	rth integral of modified Bessel function of order 0;
$L(v)$,	perimeter function defined in Section 4;
P,	perimeter [m];
Q,	heat flux [W m^{-2}];
S,	region of integration;
$T(x, y, z, t)$, $T(x, z, t)$,	time-dependent temperatures [K];
$T(x,y,z)$, $T(x,z)$,	temperature amplitudes [K];
ΔT,	temperature difference [K];
T_1, T_2,	particular values of temperature amplitude [K];
U,	thermal conductance [W m^{-2} K^{-1}];
a,	$(i\Omega/\kappa)^{1/2}$ [m^{-1}];
c,	$A/(P/4)^2$;
d,	distance [m];
$f(x)$,	function defined in Section 3.4;
$g(\omega_1,\omega_2)$, $g(\omega)$,	defined in Sections 2.1 and 2.2;
$g(x,y)$,	function defined by equation (A1);
$h(\omega_1,\omega_2)$,	function defined in Section 5;
k,	thermal conductivity [W m^{-1} K^{-1}];
$p(x)$,	function defined by equation (A3);
r,	polar coordinate;
t,	time [s];
u,v,	dummy variables;
x,y,z,	space coordinates.

Greek symbols

Φ,	integrated heat flux, [W in 3-dim. W m^{-1} in 2-dim.];
$\Psi(x)$,	heat flux at x [W m^{-2}];
Ω,	angular frequency [s^{-1}];
α,β,	particular values of x and y coordinates [m];
γ,	$\alpha-\beta$ [m];
δ,	$\alpha+\beta$ [m];
$\delta(x)$,	delta function;
ε,	wall thickness parameter [m];
θ,	polar coordinate;
κ,	diffusivity [m^2 s^{-1}];
ρ,	polar coordinate;
ϕ,	polar coordinate;
ω,ω_1,ω_2,	Fourier transform parameters.

1. INTRODUCTION

AN ASSESSMENT of the space conditioning energy requirements of a proposed building design is now considered to be an important part of the design process. An array of complex computer-based analysis techniques is being produced to meet this need. An

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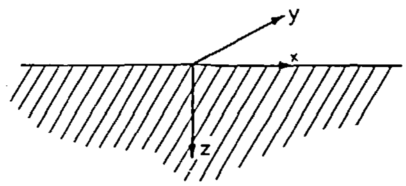

FIG. 1. The semi-infinite solid $z \geqslant 0$.

important part of such techniques is the determination of the building thermal performance, that is an analysis of time-dependent heat flows in and out of the building throughout the year.

A given building element (wall, roof, window, etc.) may be viewed as a succession of homogeneous slabs of finite thickness in contact with each other, heat flow being assumed to occur only in a direction perpendicular to the faces of the slab. Carslaw and Jaeger [1] deal extensively with such problems. This assumption of 1-dim. heat flow is perfectly reasonable for most building elements, but it is patently false for elements in contact with the ground, such as a concrete slab-on-ground or a basement-in-ground, where 2 or 3-dim. heat flow is inevitable.

The problem is essentially one of heat flow within a semi-infinite space with known surface temperature distribution (Fig. 1). Carslaw and Jaeger (ref. [1] Section 14.9) provide a very general solution to this type of problem, but its practical application to heat flow under buildings is difficult and it remains one of the less tractable areas of building heat transfer analysis. Let us then briefly review some previous attempts at estimation of ground heat flow under a building, that is, estimation of the total surface heat flow within the building floor area as a function of temperatures at the surface within the building and external to it.

The steady-state 2-dim. case is relatively simple. Macey [2] estimated the steady-state total heat flow through the floor of a structure resting on the ground and with walls of finite thickness. His application here was to the estimation of heat loss through the floor of a kiln, furnace or drier. Denoting this total heat flow per unit area of floor by Q, Macey showed that for a constant temperature difference ΔT between the surface temperatures inside and outside the building,

$$Q = \frac{2kB\Delta T}{\pi d} \tanh^{-1}(d/D) \qquad (1)$$

where $2d$ is the separation of the walls, $2(D - d)$ is the wall thickness, k is the conductivity of the ground and B is a constant depending on the shape of the floor, varying from 1.6 for a square floor down to 1 for an infinitely long floor (see, however, Section 3.3).

Billington [3] examined the applicability of Macey's expression to buildings, using a network analyser (electrical analogue) to examine some more complicated and realistic situations. He also compared his results with measurements taken by other workers [4, 5] and confirmed the applicability of equation (1) to the estimation of steady-state heat flow through ground floors. Since the steady-state 1-dim. heat flow through a normal building element is proportional to the temperature difference across the element, the constant of proportionality being the overall thermal conductance or transmittance U for the element, then equation (1) allows us to define a 1-dim. equivalent U value for the ground as

$$U = \frac{2kB}{\pi d} \tanh^{-1}(d/D).$$

Such values have remained until the present as the basis for the conventional method of estimating heat flows through ground floors in the U.K. [6].

While considerations of the steady state are adequate for an estimation of long-term average heat flows, the non-steady state is important in the evaluation of intermittent phenomena, that is estimating energy requirements for an intermittently occupied building.

Billington [7] considered a simplified analytical model of 1-dim. heat flow into a semi-infinite ground, with the surface temperature varying periodically in time. (In effect, he chose to neglect edge losses in the non-steady state.) This analysis enabled the quantity and time of occurrence of maximum heat flow during the cycle to be evaluated. A similar solution, though for transient rather than periodic surface conditions, has recently been put forward by Kaushik and Srivastava [8]. The difficulty of evaluation of the exact time-dependent, 3-dim. solution of Carslaw and Jaeger (ref. [1], Section 14.9), in practical situations has deterred most workers from using it. Lachenbruch [9] considered the 3-dim. case with periodic surface conditions, and using Green's functions developed a complex analytical expression for the temperature in terms of a double integral. Vuorelainen [10, 11] considered a 2-dim. situation with a transient surface condition and a rectangular floor plan, but in application to the estimation of heat flows through ground floors [12] only the steady-state solution appears to have been used. More recently, Muncey and Spencer [13] considered the surface to be covered by an array of equally spaced identical rectangular slabs, with the surface temperature condition periodic in time. In this way the problem could be solved using Fourier analysis and the solution evaluated relatively easily. They then furthered the idea of Billington [3] by using this solution to derive expressions for "one-dimensional equivalent" non-steady state thermal parameters for the ground. The work of Muncey and Spencer [13] must be viewed as a major step forward in the estimation of non-steady state heat flows through a ground floor.

The current work re-examines the problem considered by Vuorelainen [10–12] and Muncey and Spencer [13]. However, the surface contains only a single floor, rectangular in shape. Using a method similar to that of Vuorelainen [10, 11], a formal exact solution for the heat flow at the surface of the floor is obtained. This heat flow is proportional to the temperature gradient at the surface, and in the following sections it is shown how this gradient may be expressed as the convolution of a Fourier transform with the known temperature distribution at the surface. The problem is thus reduced to the evaluation firstly of the appropriate Fourier transform (Appendix A) and then of the convolution integral; any explicit evaluation of the temperature throughout the semi-infinite solid is avoided. For non-steady state 2-dim. flow, it is shown that the total heat flow may be split into two components, the edge component and the downward or 1-dim. component. This splitting enables the 2-dim. solution to be adapted to provide an accurate approximation for the 3-dim. non-steady state heat flow into the ground, and it is shown that the edge or perimeter component is only significant at frequencies considerably less than the diurnal (1 cycle/day). Finally, an explicit analytical expression for the 3-dim. steady-state heat flow from a rectangular floor is obtained, and it is found to be in excellent agreement with the graphical results of Vuorelainen [12].

2. FORMAL SOLUTIONS OF THE DIFFUSION EQUATION

2.1. *Three dimensions*

Let us model the ground by a semi-infinite solid $z \geqslant 0$ (Fig. 1), and assume that the temperature T in this region is oscillatory in time, with angular frequency Ω; that is,

$$T(x,y,z,t) = T(x,y,z)e^{i\Omega t}.$$

If the temperature distribution $T(x,y,0)$ at the surface is a known function, then a formal solution of the diffusion equation

$$\nabla^2 T(x,y,z,t) = \frac{1}{\kappa}\frac{\partial T(x,y,z,t)}{\partial t}$$

is

$$T(x,y,z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega_1 x - i\omega_2 y} \times e^{-z(\omega_1^2+\omega_2^2+a^2)^{1/2}} g(\omega_1,\omega_2)\,d\omega_1\,d\omega_2$$

where $a = (i\Omega/\kappa)^{1/2}$, κ is the diffusivity, and

$$g(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\omega_1 x + i\omega_2 y} T(x,y,0)\,dx dy.$$

The total flux into the ground from a region S is given by

$$\Phi = -k\int_S\int -\frac{\partial T(x,y,z)}{\partial z}\bigg|_{z=0} dx dy \qquad (2)$$

where k is the conductivity, and

$$\frac{\partial T}{\partial z}\bigg|_{z=0} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -(\omega_1^2+\omega_2^2+a^2)^{1/2} \times e^{-i\omega_1 x - i\omega_2 y} g(\omega_1,\omega_2)\,d\omega_1\,d\omega_2 \qquad (3)$$

or

$$\frac{\partial T}{\partial z}\bigg|_{z=0} = \frac{1}{(2\pi)^2} F_{x,y}[-(\omega_1^2+\omega_2^2+a^2)^{1/2}] * T(x,y,0) \qquad (4)$$

by the convolution theorem. ($F_{x,y}$ denotes a Fourier transform with respect to x and y and * denotes the convolution operation.) From Appendix A, we have

$$F_{x,y}[-(\omega_1^2+\omega_2^2+a^2)^{1/2}] = \frac{2\pi e^{-ar}}{r^3}(1+ar) \qquad (5)$$

where

$$r = (x^2+y^2)^{1/2}.$$

In principle, then, equations (2), (4) and (5) can be used to find the total heat flux into the ground from a region (say a concrete slab-on-ground) if we know the temperature distribution over the entire surface. However, in practice even the simplest non-trivial temperature distribution yields an intractable convolution integral.

The method described here can also be used when the flux instead of the temperature amplitude is specified at the surface, but not, in general, for mixed temperature–flux conditions. This means that where a flux condition might be natural (for example, where a wall overlies a surface), it is necessary to infer an approximate temperature distribution in this region, as is done in the following sections.

2.2. *Two dimensions*

For two dimensions we suppress the y-coordinate and model the ground by a semi-infinite plane $z \geqslant 0$, and write

$$T(x,z,t) = T(x,z)e^{i\Omega t},$$

with $T(x,0)$ a known function. The solution is then

$$T(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} e^{-z(\omega^2+a^2)^{1/2}} g(\omega)\,d\omega,$$

where

$$g(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} T(x,0)\,dx.$$

As before, the total flux into the ground from a region $\alpha \leqslant x \leqslant \beta$ is

$$\Phi = -k \int_{\alpha}^{\beta} \left.\frac{\partial T(x,z)}{\partial z}\right|_{z=0} \mathrm{d}x \tag{6}$$

where

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} -(\omega^2 + a^2)^{1/2} \mathrm{e}^{-\mathrm{i}\omega x} g(\omega)\,\mathrm{d}\omega \tag{7}$$

or

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = \frac{1}{2\pi} F_x[-(\omega^2 + a^2)^{1/2}]^* T(x,0). \tag{8}$$

From Appendix A, we have

$$F_x[-(\omega^2 + a^2)^{1/2}] = 2aK_1(a|x|)\left[\frac{1}{|x|} - 2\delta(x)\right] \tag{9}$$

where $K_\nu(z)$ is the modified Bessel function.

As an alternative to using equations (6), (8) and (9), one may substitute equation (7) into equation (6) and perform the x-integral first. The result for the total flux is then

$$\Phi = -\frac{k}{2\pi \mathrm{i}}\left[\int_{-\infty}^{\infty} -\frac{(\omega^2 + a^2)^{1/2}}{\omega} \mathrm{e}^{-\mathrm{i}\omega\alpha} g(\omega)\,\mathrm{d}\omega + \int_{-\infty}^{\infty} \frac{(\omega^2 + a^2)^{1/2}}{\omega} \mathrm{e}^{-\mathrm{i}\omega\beta} g(\omega)\,\mathrm{d}\omega\right]$$

or

$$\Phi = -\frac{k}{2\pi \mathrm{i}}\left[F_\alpha\left(\frac{(\omega^2 + a^2)^{1/2}}{\omega}\right)^* T(\alpha,0) - F_\beta\left(\frac{(\omega^2 + a^2)^{1/2}}{\omega}\right)^* T(\beta,0)\right] \tag{10}$$

by the convolution theorem. From Appendix A, we have

$$F_\alpha\left[\frac{(\omega^2 + a^2)^{1/2}}{\omega}\right] = 2a\mathrm{i}\,\mathrm{sgn}(\alpha)(K_1(a|\alpha|) - Ki_1(a|\alpha|) + \pi/2), \tag{11}$$

and similarly for F_β, where

$$\mathrm{sgn}(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0, \end{cases}$$

and the repeated Bessel function integrals are defined as usual by

$$Ki_r(x) = \int_x^\infty Ki_{r-1}(t)\,\mathrm{d}t;\ Ki_1(x) = \int_x^\infty K_0(t)\,\mathrm{d}t.$$

Combining equations (10) and (11), the total flux is given by

$$\Phi = \frac{ka}{\pi}\{\mathrm{sgn}(\beta)[(\pi/2) + K_1(a|\beta|) - Ki_1(a|\beta|)]^* T(\beta, 0) - \mathrm{sgn}(\alpha)[(\pi/2) + K_1(a|\alpha|) - Ki_1(a|\alpha|)]^* T(\alpha, 0)\}. \tag{12}$$

In general, equation (12) is simpler to use than equations (6)–(9). However, for the purposes of adapting these 2-dim. results to the 3-dim. case, some results using equations (6)–(9) will also be used.

3. SOME SPECIAL TWO-DIMENSIONAL RESULTS

3.1. *The inclined-step temperature distribution*

Consider the temperature distribution shown in Fig. 2, and given by

$$T(x,0) = \begin{cases} T_1, -\infty < x \leqslant -\varepsilon, \\ (T_2 - T_1)x/2\varepsilon + (T_1 + T_2)/2, |x| \leqslant \varepsilon \\ T_2, \varepsilon \leqslant x < \infty. \end{cases} \tag{13}$$

This could represent for example the interior of a building with temperature amplitude T_1, a wall of thickness 2ε, and the exterior with temperature amplitude T_2.

The assumed temperature amplitude distribution must have gradients that are finite (although not necessarily continuous) in order to avoid singularities in the calculated surface heat flux. The assumption of a linear change from T_1 to T_2 made here is the simplest available. There would be no difficulty in choosing other functions to model, for example, a composite wall.

Let

$$I(x) = \left.\frac{\partial T(x,z)}{\partial z}\right|_{z=0}$$

and define $B(z)$ by

$$B(z) \equiv z[Ki_1(z) - K_1(z)].$$

Then, using equations (8) and (9), we have

3.1.1. *If* $|x| \geqslant \varepsilon$.

$$I(x) = \left\{\frac{(T_1 - T_2)\,\mathrm{sgn}(x)}{2\pi\varepsilon}\right\}\{K_0[a(|x| - \varepsilon)] - K_0[a(|x| + \varepsilon)]$$

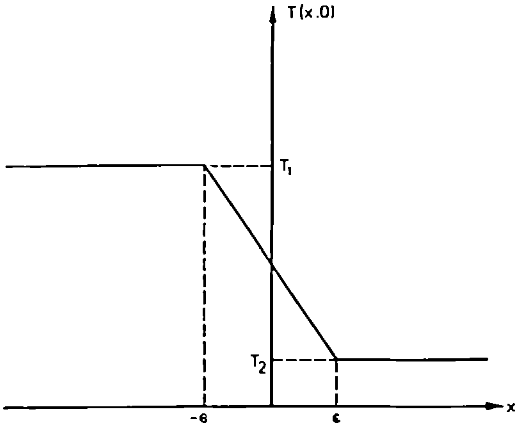

FIG. 2. The inclined-step temperature distribution.

$$- B[a(|x| + \varepsilon)] + B[a(|x| - \varepsilon)]\}$$

$$+ \frac{a}{2}[(T_1 - T_2)\operatorname{sgn}(x) - (T_1 + T_2)]. \qquad (14)$$

3.1.2. *If* $|x| \leqslant \varepsilon$.

$$I(x) = \frac{(T_1 - T_2)}{2\pi\varepsilon}\{K_0[a(\varepsilon - x)] - K_0[a(\varepsilon + x)]$$

$$- B[a(\varepsilon + x)] + B[a(\varepsilon - x)]\}$$

$$+ \frac{a}{2}\left[\frac{x}{\varepsilon}(T_1 - T_2) - (T_1 + T_2)\right]. \qquad (15)$$

3.2. *The double inclined-step temperature distribution*

Consider the temperature distribution shown in Fig. 3, and given by

$$T(x, 0) = \begin{cases} T_1, \ |x| \leqslant \beta, \\ T_1 + \dfrac{(T_1 - T_2)}{2\varepsilon}(\beta - |x|), \\ \qquad \beta \leqslant |x| \leqslant \beta + 2\varepsilon, \\ T_2, \ |x| \geqslant \beta + 2\varepsilon. \end{cases} \qquad (16)$$

This could represent the same situation as in the previous section except that both walls of the building, each of thickness 2ε, are now to be considered.

For this temperature distribution it is convenient to use equation (12) to calculate the total flux in the region $|x| \leqslant \beta$. The result is

$$\Phi = \frac{k(T_1 - T_2)}{\pi a \varepsilon}\left\{\frac{\pi}{4} - Ki_1(2a\varepsilon) + Ki_3(2a\varepsilon)\right.$$

$$- Ki_1(2a\beta) + Ki_3(2a\beta) + Ki_1[2a(\beta + \varepsilon)]$$

$$\left.- Ki_3[2a(\beta + \varepsilon)]\right\} + 2\beta a k T_1. \qquad (17)$$

For completeness, equations (8) and (9) have also been used to calculate the flux at any point. The results are given in Appendix B.

3.3. *The steady-state limit*

We note that equation (17) may be used to obtain the steady-state heat flux for the temperature distribution given by equation (16). To do this we need the limit of equation (17) as a tends to zero (that is, $\Omega \to 0$). This can be calculated by using the following small $-|z|$ expansion of the Bessel function integrals:

$$\frac{\pi}{4} - Ki_1(z) + Ki_3(z) \simeq -z \ln z + \ldots$$

The total flux for the steady-state case is then easily found to be

$$\Phi(\Omega = 0) = \frac{2k(T_1 - T_2)}{\pi}\left\{\ln\left[\frac{\beta + \varepsilon}{\varepsilon}\right] + \frac{\beta}{\varepsilon}\ln\left[\frac{\beta + \varepsilon}{\beta}\right]\right\}.$$

Note that this result is not the same as that of Macey [2] for an infinitely long floor. The reason for the discrepancy lies in his use of the temperature gradient appropriate to an infinitely thin wall rather than the exact expression for walls of finite thickness.

3.4. *Interaction between opposite walls*

Let us examine the extent to which the solution of Section 3.1 approximates that of Section 3.2, that is, the extent to which opposite walls interact.

Consider the flux $\Psi(x)$ at any point due to the temperature distribution given by equation (13), when $x \leqslant -\varepsilon$

$$\Psi(x) = -kI(x) = \frac{k(T_1 - T_2)}{2\pi\varepsilon}f(x) + akT_1 \quad (18)$$

where from equation (14),

$$f(x) = K_0[a(|x| - \varepsilon)] - K_0[a(|x| + \varepsilon)]$$
$$- B[a(|x| + \varepsilon)] + B[a(|x| - \varepsilon)].$$

Noting that

$$f(x) \to 0 \text{ as } x \to -\infty$$

and

$$|f(x)| \to \infty \text{ as } x \to -\varepsilon,$$

then in the above expression for $\Psi(x)$ the first term represents the flux due to the presence of the wall at $x = -\varepsilon$, and the second term represents the 1-dim. downward flux at any point, akT_1, due to a constant temperature amplitude of T_1 everywhere. [This follows immediately from equation (7) since $g(\omega)$ is a delta function if $T(x, 0)$ is constant.]

The total flux in the region $x < -\varepsilon$ caused by the presence of the wall is therefore obtained by integrat-

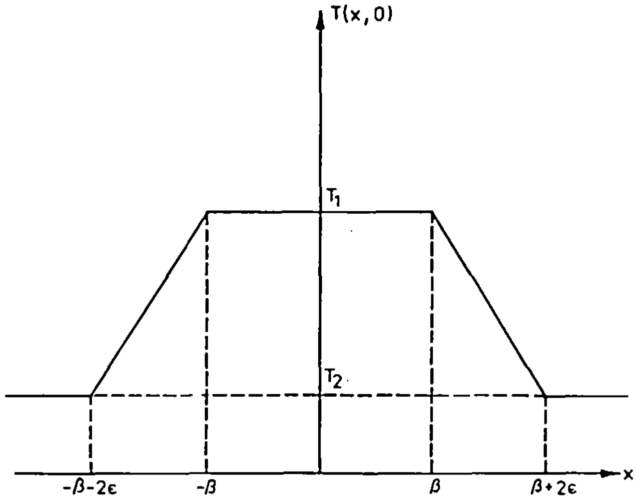

FIG. 3. The double inclined-step temperature distribution.

ing the first term in equation (18) from $-\infty$ to $-\varepsilon$. The result is

$$\frac{k(T_1 - T_2)}{2\pi\varepsilon}\int_{-\infty}^{-\varepsilon} f(x)\,dx = \frac{k(T_1 - T_2)}{2\pi a\varepsilon} \times \left[\frac{\pi}{4} - Ki_1(2a\varepsilon) + Ki_3(2a\varepsilon)\right]. \quad (19)$$

Suppose now we have two walls separated by a distance 2β, with a temperature distribution given by equation (16). We postulate that if 2β is sufficiently large, then the total flux in the region between the walls can be obtained by neglecting the effect of one wall on the other. Thus, using equations (18) and (19), the total flux in this approximation should be

$$\Phi = 2\int_{-\infty}^{-\varepsilon} \frac{k(T_1 - T_2)}{2\pi\varepsilon} f(x)\,dx + \int_{-\beta}^{\beta} kaT_1\,dx$$

or

$$\Phi = \frac{k(T_1 - T_2)}{\pi a\varepsilon}\left[\frac{\pi}{4} - Ki_1(2a\varepsilon) + Ki_3(2a\varepsilon)\right] + 2\beta ak\,T_1. \quad (20)$$

In appendix C, the accuracy of equation (20) is checked by comparing it with the exact solution given by equation (17). From this comparison we conclude that for realistic wall separations, and for periods of up to 1 year, the effect of the opposite wall can be neglected to a high degree of accuracy.

4. THE PERIMETER FUNCTION

In the previous section it was established that in 2-dim. flow the walls can be treated independently of each other when calculating the total flux in the region between them. This result will now be adapted to the more realistic 3-dim. case.

Consider a slab of arbitrary rectilinear shape, with area A, perimeter P, interior temperature amplitude T_1, exterior temperature amplitude T_2 and surrounded by a wall of thickness 2ε. We will assume that along any line perpendicular to the walls the conclusions of the previous sections hold, viz. that the flux into the ground consists of a component representing the 1-dim. flow vertically down, which at any point is given by the second term in equation (18), plus a component representing the curved-path flows under the wall. The flux resulting from this type of flow can be calculated for each position along the perimeter independently of the presence of an opposite wall (provided it is sufficiently far away, see Appendix C), and is given by equation (19). Thus the total flux through the slab, using equations (18) and (19), is given by

$$\Phi = \frac{kP(T_1 - T_2)}{2\pi a\varepsilon}\left[\frac{\pi}{4} - Ki_1(2a\varepsilon) + Ki_3(2a\varepsilon)\right] + akAT_1, \quad (21)$$

if corner effects are neglected.

Define a 'perimeter function' $L(v)$, given by

$$L(v) = \frac{1}{\pi v}\left[\frac{\pi}{4} - Ki_1(v) + Ki_3(v)\right].$$

The total flux as a function of time is then

$$Re\{[kP(T_1 - T_2)L(2a\varepsilon) + akAT_1]\,e^{i\Omega t}\},$$

and $L(v)$ can easily be calculated for various values of $|v| = 2\varepsilon(\Omega/\kappa)^{1/2}$. To determine the relative importance of the perimeter term $kP(T_1 - T_2)L(2a\varepsilon)$ and the area term $akAT_1$ at various frequencies, let us take a typical

Table 1.

Period (days)	$PL(2a\varepsilon)$ (m)	aA (m)
1	2.92 − 2.82i	889(1 + i)
30	14.5 − 7.49i	162(1 + i)
365	28.2 − 9.17i	46.5(1 + i)

floor slab of dimensions 10 × 10 m, with a wall thickness of 0.2 m. Suppose for simplicity that the internal temperature amplitude is 10 K and the external amplitude is 20 K, so that we need only compare the relative sizes of $PL(2a\varepsilon)$ and aA. The values of the perimeter term and the area term have been calculated for three values of the period (1 day, 30 days, and 365 days) using the value $\kappa = 4.6 \times 10^{-7}\,\mathrm{m^2\,s^{-1}}$ for average soil [1]. The results are given in Table 1.

Clearly the perimeter component only begins to become significant at periods much greater than one day, i.e. in the region of steady-state behaviour.

5. THREE-DIMENSIONAL STEADY STATE HEAT FLOW

The results of the previous section are not valid for the 3-dim. case in the steady state. This case has been treated by Vuorelainen [10–12], who obtained the total heat flow Φ from a rectangular floor, in the form of a double integral. This was evaluated numerically and the results presented in the form of nomographs. In this section we will use a similar method to obtain an exact expression for Φ.

Consider a rectangular slab in the $z = 0$ plane with dimensions $2\alpha, 2\beta$ (Fig. 4). From equation (2), the total heat flow into the ground is given by

$$\Phi = -k \int_{-\alpha}^{\alpha} \int_{-\beta}^{\beta} \left.\frac{\partial T}{\partial z}\right|_{z=0} \mathrm{d}x\mathrm{d}y$$

where $\partial T/\partial z$ at $z = 0$ is given by equation (3) with $a = 0$ for steady-state conditions. Performing the x, y integrals in the above equation gives

$$\Phi = -\frac{k}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_1 \omega_2} (\mathrm{e}^{\mathrm{i}\omega_1\alpha} - \mathrm{e}^{-\mathrm{i}\omega_1\alpha}) \times (\mathrm{e}^{\mathrm{i}\omega_2\beta} - \mathrm{e}^{-\mathrm{i}\omega_2\beta})\, g(\omega_1, \omega_2)\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2$$

or, writing $T(x, y, 0)$ as $T(x, y)$,

$$\begin{aligned}\Phi = -\frac{k}{4\pi^2}\{&F_{-\alpha,-\beta}[h(\omega_1,\omega_2)] * T(-\alpha, -\beta) \\ &+ F_{\alpha,\beta}[h(\omega_1,\omega_2)] * T(\alpha,\beta) \\ &- F_{-\alpha,\beta}[h(\omega_1,\omega_2)] * T(-\alpha,\beta) \\ &- F_{\alpha,-\beta}[h(\omega_1,\omega_2)] * T(\alpha,-\beta)\} \qquad (22)\end{aligned}$$

by the convolution theorem, where

$$h(\omega_1,\omega_2) = \frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_1 \omega_2}.$$

In Appendix A it is shown that

$$F_{x,y} \frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_1 \omega_2} = -\frac{2\pi(x^2 + y^2)^{1/2}}{xy}.$$

Using this expression in equation (22), and noting that we can choose axes so that $T(x, y) = T(\pm x, \pm y)$, the total flux is given by

$$\Phi(\alpha, \beta) = \frac{2k(\alpha^2 + \beta^2)^{1/2}}{\pi\alpha\beta} * T(\alpha, \beta).$$

Explicitly,

$$\Phi(\alpha, \beta) = \frac{2k}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[(\alpha - u)^2 + (\beta - v)^2]^{1/2}}{[(\alpha - u)(\beta - v)]} \times T(u, v)\, \mathrm{d}u\mathrm{d}v. \qquad (23)$$

To avoid singularities at $u = \alpha$ and $v = \beta$, a wall of finite thickness 2ε is required. Assume that within the walls the surface temperature is 0, and that the temperature falls linearly from 0 to −1 over the distance 2ε. In the steady state, all surface temperature distributions with the same inside–outside temperature difference will give the same result.

Referring to Fig. 4, $T(x, y)$ is given as

$$T(x,y) = \begin{cases} 0, & |x| \leqslant \alpha,\ |y| \leqslant \beta \\ -1, & |x| \geqslant \alpha + 2\varepsilon \text{ and/or } |y| \geqslant \beta + 2\varepsilon \\ \dfrac{\beta - y}{2\varepsilon}, & \beta \leqslant y \leqslant \beta + 2\varepsilon,\ |x| \leqslant y + \alpha - \beta \ \text{(Region I)} \\ \dfrac{\alpha - x}{2\varepsilon}, & \alpha \leqslant x \leqslant \alpha + 2\varepsilon,\ |y| \leqslant x + \beta - \alpha \ \text{(Region II)} \\ \dfrac{\beta + y}{2\varepsilon}, & -\beta - 2\varepsilon \leqslant y \leqslant -\beta,\ |x| \leqslant -y + \alpha - \beta \ \text{(Region III)} \\ \dfrac{\alpha + x}{2\varepsilon}, & -\alpha - 2\varepsilon \leqslant x \leqslant -\alpha,\ |y| \leqslant -x + \beta - \alpha \ \text{(Region IV)} \end{cases} \qquad (24)$$

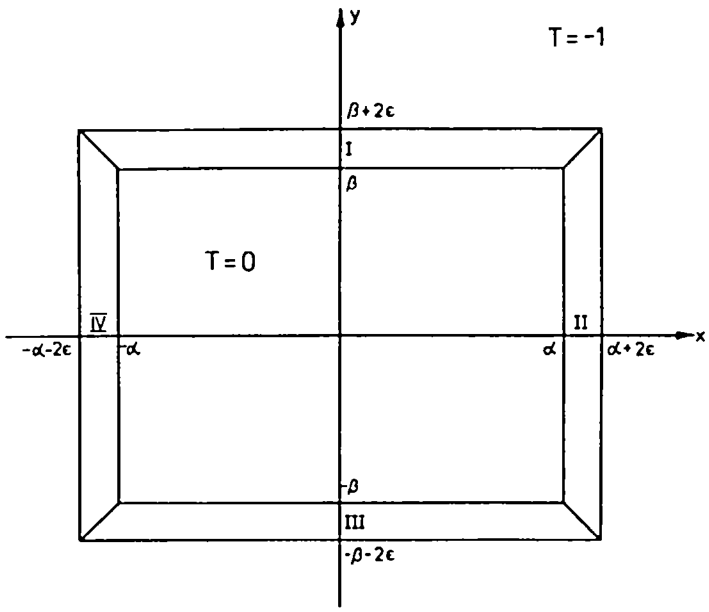

FIG. 4. The surface temperature distribution for 3-dim. steady-state heat flow.

Combining equations (23) and (24) gives

$$\Phi(\alpha,\beta) = \frac{2k}{\pi} G(\alpha,\beta) \qquad (25)$$

where

$$G(\alpha,\beta) = \left(2 + \frac{\alpha+\beta}{\varepsilon}\right)(\alpha+\varepsilon)^2 + (\beta+\varepsilon)^2]^{1/2} - (\sqrt{2})\left(1 + \frac{\alpha}{2\varepsilon}\right)[\alpha^2 + (\alpha+2\varepsilon)^2]^{1/2}$$

$$- (\sqrt{2})\left(1 + \frac{\beta}{2\varepsilon}\right)[\beta^2 + (\beta+2\varepsilon)^2]^{1/2} - \frac{\alpha+\beta}{\varepsilon}(\alpha^2+\beta^2)^{1/2}$$

$$+ \frac{\alpha^2+\beta^2}{\varepsilon}\{1 + (\sqrt{2})\ln[(\sqrt{2}) - 1]\} + 2\varepsilon\{(\sqrt{2}) + \ln[(\sqrt{2}) - 1]\}$$

$$- \frac{(\sqrt{2})\gamma^2}{\varepsilon}\ln\left[\frac{[\gamma^2 + (\delta+2\varepsilon)^2]^{1/2} + \delta + 2\varepsilon}{(\gamma^2+\delta^2)^{1/2} + \delta}\right] + \frac{\gamma^2 - (\beta+\varepsilon)^2}{\varepsilon}\ln\left[\frac{[(\alpha+\varepsilon)^2 + (\beta+\varepsilon)^2]^{1/2} + \beta + \varepsilon}{\alpha+\varepsilon}\right]$$

$$+ \frac{\gamma^2 - (\alpha+\varepsilon)^2}{\varepsilon}\ln\left[\frac{[(\alpha+\varepsilon)^2 + (\beta+\varepsilon)^2]^{1/2} + \alpha + \varepsilon}{\beta+\varepsilon}\right] + \frac{\alpha(2\beta-\alpha)}{\varepsilon}\left[\ln\frac{(\alpha^2+\beta^2)^{1/2} + \beta}{\alpha}\right]$$

$$+ \frac{\beta(2\alpha-\beta)}{\varepsilon}\ln\left[\frac{(\alpha^2+\beta^2)^{1/2} + \alpha}{\beta}\right] - \frac{\alpha^2-\varepsilon^2}{\varepsilon}\ln\left[\frac{[\varepsilon^2 + (\alpha+\varepsilon)^2]^{1/2} + \varepsilon}{\alpha+\varepsilon}\right]$$

$$- \frac{\beta^2-\varepsilon^2}{\varepsilon}\ln\left[\frac{[\varepsilon^2 + (\beta+\varepsilon)^2]^{1/2} + \varepsilon}{\beta+\varepsilon}\right]$$

$$+ (2\alpha+\varepsilon)\ln\left[\frac{[\varepsilon^2 + (\alpha+\varepsilon)^2]^{1/2} + \alpha + \varepsilon}{\varepsilon}\right]$$

$$+ (2\beta+\varepsilon)\ln\left[\frac{[\varepsilon^2 + (\beta+\varepsilon)^2]^{1/2} + \beta + \varepsilon}{\varepsilon}\right]$$

$$+ \frac{(\sqrt{2})\alpha^2}{\varepsilon}\ln\left[\frac{[\alpha^2 + (\alpha+2\varepsilon)^2]^{1/2} + \alpha + 2\varepsilon}{\alpha}\right]$$

$$+ \frac{(\sqrt{2})\beta^2}{\varepsilon}\ln\left[\frac{[\beta^2 + (\beta+2\varepsilon)^2]^{1/2} + \beta + 2\varepsilon}{\beta}\right]$$

and $\gamma = \alpha - \beta$, $\delta = \alpha + \beta$.

The 2-dim. result of Section 3.3 can be obtained by dividing equation (25) by 2α and letting $\alpha \to \infty$. Note also that great simplification results in the case where $\alpha \gg \varepsilon$, $\beta \gg \varepsilon$. This, of course, is justified in most cases and gives

$$\simeq \frac{4k}{\pi}\left[\alpha \ln\left(\frac{2\alpha}{\varepsilon}\right) + \beta \ln\left(\frac{2\beta}{\varepsilon}\right) + 2(\alpha^2+\beta^2)^{1/2} - \alpha - \beta - \beta \ln \frac{[(\alpha^2+\beta^2)^{1/2}+\beta]}{\alpha} - \alpha \ln \frac{[(\alpha^2+\beta^2)^{1/2}+\alpha]}{\beta}\right]. \tag{26}$$

Equations (25) or (26) can be compared with the graphical results of Vuorelainen [12] for slabs of various dimensions, and wall thicknesses of 0.25, 0.30 and 0.35 m. Detailed comparisons were made using equation (26), and in all cases the discrepancies were less than 4%, which is close to the accuracy with which values can be read from the graphs of ref. [12]. Even these small discrepancies disappeared when the exact expression given by equation (25) was used for the comparison.

If the slab shape is not rectangular but rectilinear, we propose, following the method suggested by Muncey and Spencer [13], that the total heat flow from the slab whose area is A and perimeter is P be set equal to the total heat flow from an equivalent rectangular slab which has the same area and perimeter. The dimensions of the equivalent rectangular slab, 2α and 2β, can be calculated from the equations

$$\alpha = \frac{1}{8}[P + (P^2 - 16A)^{1/2}]$$

$$\beta = \frac{1}{8}[P - (P^2 - 16A)^{1/2}].$$

With these values of α and β, equations (25) or (26) may then be used to find Φ.

6. CONCLUSIONS

The results of the previous sections, when taken together, provide a relatively complete and practical solution to the problem of 3-dim. heat flow into the ground underneath a rectilinear slab. Assuming harmonic time dependence with angular frequency Ω, then the solution for the harmonic amplitude Φ of the total heat flux at the surface may be expressed as a function of Ω in the following manner, provided that the assumptions of Section 4 hold:

(1) For $\Omega > 1$ cycle per day, the heat flow is essentially 1-dim., that is, perpendicular to the slab surface, and the solution has an appropriately simple form, given by

$$\Phi \simeq akAT_1.$$

(2) For Ω between 1 cycle/year and 1 cycle/day, the heat flow is truly 3-dim., and to a very good approximation consists of the curved-path flows around the edge or perimeter of the slab plus the 1-dim. flow applicable to (1), as shown in equation (21).

(3) For $\Omega = 0$ (the steady state), we have an exact expression for Φ in the case of a rectangular slab, given by equation (25). For a non-rectangular slab, a sufficient approximation exists [13] (see Section 5).

Note that we do not have a solution when Ω is between 0 and 1 cycle/year. However, the frequency 1 cycle/year is sufficiently close to the steady-state for this matter to be of no practical concern.

All the solutions presented here are simple in form and mainly analytical. Furthermore, the results of Sections 4 and 5 can be used to obtain solutions for more complicated boundary conditions: for example, a constant temperature over the slab and a (different) constant temperature plus a harmonic time-dependent component outside the slab. The only numerical evaluation required is for the two repeated Bessel function integrals $Ki_1(z)$ and $Ki_3(z)$, which are well-known functions.

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APPENDIX A
SOME FOURIER TRANSFORMS

1. *Calculation of* $F_{x,y}[-(\omega_1^2 + \omega_2^2 + a^2)^{1/2}]$

Let

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i\omega_1 x + i\omega_2 y}}{(\omega_1^2+\omega_2^2+a^2)^{1/2}}\, d\omega_1\, d\omega_2, \tag{A1}$$

and change to polar coordinates by writing $\omega_1 = \rho\cos\theta$, $\omega_2 = \rho\sin\theta$, so that

$$g(x,y) = \int_0^\infty \int_{-\pi}^{\pi} \frac{\rho e^{i(x\rho\cos\theta + y\rho\sin\theta)}}{(\rho^2 + a^2)^{1/2}} d\rho\, d\theta.$$

If we make the further transformation $x = r\cos\phi, y = r\sin\phi$, then

$$g(x,y) = \int_0^\infty \left[\int_0^\pi d\theta\, e^{ir\rho\cos(\theta-\phi)} + \int_0^\pi d\theta\, e^{ir\rho\cos(\theta+\phi)}\right]\left[\frac{\rho d\rho}{(\rho^2+a^2)^{1/2}}\right]$$

or

$$g(x,y) = 2\pi \int_0^\infty \left[\frac{\rho J_0(r\rho)}{(\rho^2+a^2)^{1/2}}\right] d\rho = \frac{2\pi e^{-ar}}{r}, \quad \text{(A2)}$$

by the use of the integral representation of the Bessel function and a table of integrals [14].
The integral we require is

$$F_{x,y}[-(\omega_1^2 + \omega_2^2 + a^2)^{1/2}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -(\omega_1^2 + \omega_2^2 + a^2)^{1/2} e^{i\omega_1 x + i\omega_2 y} d\omega_1\, d\omega_2.$$

It is easy to see from equation (A1) that

$$F_{x,y}[-(\omega_1^2 + \omega_2^2 + a^2)^{1/2}] = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} - a^2 g.$$

Hence, using equation (A2), we obtain

$$F_{x,y}[-(\omega_1^2 + \omega_2^2 + a^2)^{1/2}] = \left[\frac{2\pi e^{-ar}}{r^3}\right](1 + ar).$$

2. *Calculation of* $F_{x,y}[(\omega_1^2 + \omega_2^2)^{1/2}/\omega_1\omega_2]$
Let us take $a = 0$ in equations (A1) and (A2). Then

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{e^{i\omega_1 x + i\omega_2 y}}{(\omega_1^2 + \omega_2^2)^{1/2}}\right] d\omega_1\, d\omega_2 = \frac{2\pi}{r},$$

where $r = (x^2 + y^2)^{1/2}$.

Now

$$g_x(x,y) = \frac{\partial g}{\partial x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{i\omega_1 e^{i\omega_1 x + i\omega_2 y}}{(\omega_1^2 + \omega_2^2)^{1/2}}\right] d\omega_1\, d\omega_2,$$

and so

$$\int_0^y g_x(x,y')\,dy' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\omega_1 e^{i\omega_1 x}(e^{i\omega_2 y} - 1)}{\omega_2(\omega_1^2 + \omega_2^2)^{1/2}}\right] d\omega_1\, d\omega_2,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\omega_1 e^{i\omega_1 x + i\omega_2 y}}{\omega_2(\omega_1^2 + \omega_2^2)^{1/2}}\right] d\omega_1\, d\omega_2,$$

since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\omega_1 e^{i\omega_1 x}}{\omega_2(\omega_1^2 + \omega_2^2)^{1/2}}\right] d\omega_1\, d\omega_2 = 0,$$

by symmetry.

Similarly:

$$\int_0^x g_y(x',y)\,dx' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\omega_2 e^{i\omega_1 x + i\omega_2 y}}{\omega_1(\omega_1^2 + \omega_2^2)^{1/2}}\right] d\omega_1\, d\omega_2.$$

Therefore,

$$\int_0^x g_y(x',y)\,dx' + \int_0^y g_x(x,y')\,dy'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_1\omega_2}\right] e^{i\omega_1 x + i\omega_2 y} d\omega_1\, d\omega_2,$$

$$= F_{x,y}\left[\frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_1\omega_2}\right]$$

But

$$g_x(x,y) = -\frac{2\pi x}{r^3},\; g_y(x,y) = -\frac{2\pi y}{r^3},$$

and so

$$F_{x,y}\left[\frac{(\omega_1^2 + \omega_2^2)^{1/2}}{\omega_1\omega_2}\right] = -\frac{2\pi(x^2 + y^2)^{1/2}}{xy}$$

3. *Calculation of* $F_x[-(\omega^2 + a^2)^{1/2}]$
We require the integral

$$F_x[-(\omega^2 + a^2)^{1/2}] = \int_{-\infty}^{\infty} -(\omega^2 + a^2)^{1/2} e^{i\omega x} d\omega$$

$$= 2\int_0^\infty -(\omega^2 + a^2)^{1/2}\cos\omega x\, d\omega.$$

Let

$$p(x) = 2\int_0^\infty \left[\frac{\cos\omega x}{(\omega^2 + a^2)^{1/2}}\right] d\omega. \quad \text{(A3)}$$

Using the integral representation [14] of the modified Bessel function $K_0(z)$, we have

$$p(x) = 2K_0(a|x|). \quad \text{(A4)}$$

The modulus sign has been inserted into the argument of the Bessel function because $p(x)$ needs to be defined for all values of x. From equation (A3), we have

$$\frac{d^2p}{dx^2} = -2\int_0^\infty (\omega^2 + a^2)^{1/2}\cos\omega x\, d\omega + 2a^2\int_0^\infty \left[\frac{\cos\omega x}{(\omega^2 + a^2)^{1/2}}\right] d\omega,$$

so that

$$F_x[-(\omega^2 + a^2)^{1/2}] = \frac{d^2p}{dx^2} - a^2 p.$$

From equation (A4)

$$\frac{dp}{dx} = -2a\,\mathrm{sgn}(x) K_1(a|x|)$$

and

$$\frac{d^2p}{dx^2} = 2a^2 K_0(a|x|) + 2aK_1(a|x|)\left[\frac{1}{|x|} - 2\delta(x)\right].$$

Hence

$$F_x[-(\omega^2+a^2)^{1/2}] = 2aK_1(a|x|)\left[\frac{1}{|x|} - 2\delta(x)\right].$$

4. *Calculation of* $F_x[(\omega^2+a^2)^{1/2}/\omega]$
Let

$$p(\alpha) = \int_{-\infty}^{\infty}\left[\frac{e^{i\omega\alpha}}{(\omega^2+\alpha^2)^{1/2}}\right]d\omega = 2K_0(a|\alpha|). \quad \text{(A5)}$$

Then

$$\int_0^x p(\alpha')d\alpha' = \frac{1}{i}\int_{-\infty}^{\infty}\left[\frac{e^{i\omega x}}{\omega(\omega^2+a^2)^{1/2}}\right]d\omega$$

since

$$\int_{-\infty}^{\infty}\left[\frac{d\omega}{\omega(\omega^2+a^2)^{1/2}}\right] = 0$$

by symmetry.

Also,

$$\begin{aligned}\frac{dp}{d\alpha} &= i\int_{-\infty}^{\infty}\left[\frac{\omega e^{i\omega x}}{(\omega^2+a^2)^{1/2}}\right]d\omega\\ &= i\int_{-\infty}^{\infty}\left[\frac{(\omega^2+a^2)^{1/2}}{\omega}\right]e^{i\omega x}d\omega\\ &\quad - ia^2\int_{-\infty}^{\infty}\left[\frac{e^{i\omega x}}{\omega(\omega^2+\alpha^2)^{1/2}}\right]d\omega\\ &= iF_x\left[\frac{(\omega^2+a^2)^{1/2}}{\omega}\right] + a^2\int_0^x p(\alpha')d\alpha',\end{aligned}$$

so that

$$F_x\left[\frac{(\omega^2+a^2)^{1/2}}{\omega}\right] = -\frac{idp}{d\alpha} + ia^2\int_0^x p(\alpha')d\alpha'.$$

But, from equation (A5),

$$\int_0^x p(\alpha')d\alpha' = \frac{2}{a}\operatorname{sgn}(\alpha)\left[\frac{\pi}{2} - Ki_1(a|\alpha|)\right],$$

so that

$$F_x\left[\frac{(\omega^2+a^2)^{1/2}}{\omega}\right] = 2ai\operatorname{sgn}(\alpha) \times\left[K_1(a|\alpha|) - Ki_1(a|\alpha) + \frac{\pi}{2}\right].$$

APPENDIX B
HEAT FLUX FOR THE DOUBLE INCLINED-STEP TEMPERATURE DISTRIBUTION

Equation (8) was calculated using the temperature distribution given by equation (16). Writing

$$I(x) = \left.\frac{\partial T}{\partial z}\right|_{z=0}$$

and defining $B(z)$ by

$$B(z) = z[Ki_1(z) - K_1(z)],$$

the results are as follows:

(a) *If* $|x| > \beta + 2\varepsilon$,

$$\begin{aligned}I(x) = \frac{(T_1 - T_2)}{2\pi\varepsilon}\{&B[a(|x|+\beta+2\varepsilon)] + B[a(|x|-\beta-2\varepsilon)]\\ &- B[a(|x|+\beta)] - B[a(|x|-\beta)]\\ &+ K_0[a(|x|+\beta+2\varepsilon)] + K_0[a(|x|-\beta-2\varepsilon)] - K_0[a(|x|+\beta)]\\ &- K_0[a(|x|-\beta)]\} - aT_2.\end{aligned}$$

(b) *If* $|x| < \beta$, (B1)

$$\begin{aligned}I(x) = \frac{(T_1 - T_2)}{2\pi\varepsilon}\{&B[a(\beta+2\varepsilon+x)] + B[a(\beta+2\varepsilon-x)]\\ &- B[a(\beta+x)] - B[a(\beta-x)] + K_0[a(\beta+2\varepsilon+x)]\\ &+ K_0[a(\beta+2\varepsilon-x)] - K_0[a(\beta+x)]\\ &- K_0[a(\beta-x)]\} - aT_1.\end{aligned}$$

(B2)

(c) *If* $\beta < |x| < \beta + 2\varepsilon$

$$\begin{aligned}I(x) = \frac{(T_1 - T_2)}{2\pi\varepsilon}\{&B[a(\beta+2\varepsilon+x)] + B[a(\beta+2\varepsilon-x)]\\ &- B[a(|x|+\beta)] - B[a(|x|-\beta)]\\ &+ K_0[a(x+\beta+2\varepsilon)]\\ &+ K_0[a(\beta+2\varepsilon-x)] - K_0[a(|x|+\beta)]\\ &- K_0[a(|x|-\beta)]\\ &+ \pi(|x|-\beta)\}.\end{aligned} \quad \text{(B3)}$$

APPENDIX C
NEGLECTING OPPOSITE WALLS

The accuracy of equation (20) can be checked by comparing it to the exact result, given by equation (17). Let

$$\Delta = \frac{\pi}{4} - Ki_1(2a\varepsilon) + Ki_3(2a\varepsilon),$$

and

$$\delta = Ki_3(2a\beta) - Ki_1(2a\beta) + Ki_1[2a(\beta+\varepsilon)] - Ki_3[2a(\beta+\varepsilon)].$$

Then, referring to equations (17) and (20), it is clear that the presence of an opposite wall has a negligible effect on the flux due to the wall under consideration if

$$|\text{Re}(\Delta)| \gg |\text{Re}(\delta)|$$

and

$$|\text{Im}(\Delta)| \gg |\text{Im}(\delta)|$$

An accurate numerical calculation has been made of Δ and δ, using the values $\kappa = 4.6 \times 10^{-7}\,\text{m}^2\,\text{s}^{-1}$ for average soil [1], $2\varepsilon = 0.2\,\text{m}$ (a typical wall thickness) and $2\beta = 3\,\text{m}$. Some representative results for various values of the period are shown in Table 2.

Table 2.

Period (days)	Re(Δ)	Im(Δ)	Re(δ)	Im(δ)
1	0.803	0.014	$<10^{-5}$	
50	0.494	0.180	0.00030	−0.00041
100	0.409	0.172	0.0017	0.00044
150	0.363	0.163	0.0021	0.0021
200	0.332	0.156	0.0017	0.0037
250	0.309	0.150	0.011	0.0049
300	0.292	0.145	0.0031	0.0059
350	0.277	0.141	−0.00052	0.0066
400	0.265	0.137	−0.0014	0.0072

From these results it is clear that for $2\beta = 3$ m and for periods of less than 1 year, the maximum error involved in ignoring the contribution of δ to equation (17) is of the order of 5%. This error increases with increasing period but decreases with increasing β (for constant ε). These conclusions are valid provided that $\varepsilon \ll \beta$.

APPLICATION DES TRANSFORMEES DE FOURIER A LA CONDUCTION THERMIQUE PERIODIQUE DANS LE SOL SOUS UN IMMEUBLE

Résumé—Des transformées de Fourier sont utilisées pour obtenir des expressions du flux de chaleur depuis une surface à température donnée dans un solide semi-infini à deux ou trois dimensions, avec toutes les grandeurs supposées périodiques dans le temps. Ces expressions sont explicitées dans le cas bidimensionnel et elles sont utilisées pour obtenir des résultats approchés dans trois dimensions, valables pour un grand domaine de fréquences. Une expression explicite exacte est obtenue pour le flux thermique, dans le cas permanent tridimensionnel, à partir d'une surface rectangulaire.

ANWENDUNG DER FOURIERTRANSFORMATION AUF DEN PERIODISCHEN WÄRMEFLUSS IN DEM BODEN UNTER EINEM GEBÄUDE

Zusammenfassung—Fouriertransformationen werden verwendet, um Ausdrücke für den Wärmestrom von einem Gebiet an der Oberfläche mit gegebener Temperatur in einen zwei- oder dreidimensionalen halbunendlichen Festkörper zu erhalten, wobei angenommen wird, daß alle Größen periodisch in der Zeit sind. Diese Ausdrücke werden explizit für den zweidimensionalen Fall ausgewertet und benutzt, um Nährungslösungen für den dreidimensionalen Fall zu erhalten, die für einen großen Frequenzbereich gültig sind. Eine exakte explizite Lösung für den dreidimensionalen stationären Wärmefluß, der von einem rechteckigen Gebiet an der Oberfläche ausgeht, wird ebenfalls erhalten.

ИСПОЛЬЗОВАНИЕ ПРЕОБРАЗОВАНИЙ ФУРЬЕ ДЛЯ РАСЧЕТА ВЕЛИЧИНЫ ПЕРИОДИЧЕСКОГО ПОТОКА ТЕПЛА В ГРУНТ ПОД ЗДАНИЕМ

Аннотация—Для определения величины теплового потока от поверхности с заданной температурой к двумерному или трехмерному полубесконечному твердому телу используются преобразования фурье в предположении, что все величины являются периодическими во времени. Получены выражения в явном виде для двумерного случая, которые используются затем для получения приближенных результатов в трехмерном случае. Последние справедливы в широком диапазоне частот. Также получено точное явное выражение для трехмерного стационарного теплового потока от прямоугольной области поверхности.